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Spectral Analysis of Finite Section Normal Integral Operators

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Some continuity and differentiability properties of the eigenvalues and eigenfunctions of finite section normal integral operators are proved. These are the extension of corresponding results for symmetric operators (R. Vittal Rao, *J. Math. Anal. Appl.* **53** (1976), 554–566; K. B. Athreya and R. Vittal Rao, to appear; R. Vittal Rao and N. Sukavanam, *J. Math. Anal. Appl.* **109** (1985), 463–471.

1. INTRODUCTION

Finite section integral operators occur in a wide range of fields such as Radiative Transfer [1], Neutron Transport Theory [2], Linearized Gas Dynamics [3], etc. The spectral properties of such operators—in particular those of the finite section Wiener–Hopf integral operators—have been extensively studied. By a finite section integral operator, we mean an operator T_t of the form

$$T_t f = \int_0^t K(x, y) \cdot f(y) \cdot dy. \tag{1.1}$$

If the kernel $K(x, y)$ is a difference kernel, i.e., if it is of the form $K(x-y)$, then it is called a finite section Wiener-Hopf (FSWH) integral operator.

For the FSWH operator, under suitable assumptions on the kernel (basically yielding self-adjointness of T_t on $L_2[0, t]$), Vittal Rao [4–7], Mullikin and Vittal Rao [8] obtained various results on the eigenvalues as functions of t. These results have recently been extended for general finite section integral operators on general spaces by Athreya and Vittal Rao [9]. A continuity property of the eigenfunctions of T_t has been studied by Vittal Rao and Sukavanam [10]. All these results refer to self-adjoint operators. A natural sequel to the study of self-adjoint operators is that of normal operators. In this paper we extend some of the above-mentioned results, to finite section normal integral operators.

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This extension is first obtained for FSWH operators which are normal and using the ideas in the methods of Athreya and Vittal Rao, the extensions are achieved for general finite section normal integral operators.

In Section 2 we briefly review the results on the eigenvalues and eigenfunctions of self-adjoint operators [4–8] and then state the results of this paper on normal operators.

In Section 3 we state and prove a result about the multiplicity of the eigenvalues to self-adjoint integral operators.

In Section 4 we give the proofs. We conclude this paper in Section 5 with some remarks.

2. Preliminaries and Statement of Results

Let T_t be a symmetric integral operator from $L_2[0, t]$ into itself defined as in (1.1). The assumptions on the kernel $K(x)$ are

(i) $K(x) = \overline{K(-x)}$, (2.1)

(ii) $K(x)$ is locally square integrable in R^1. (2.2)

Since T_t is a symmetric and completely continuous operator on $L_2[0, t]$, its spectrum consists only of at most a countable number of real eigenvalues with zero as the only possible limit point. Let $\{\lambda_n(t)\}_{n=1}^{\infty}$ and $\{\mu_n(t)\}_{n=1}^{\infty}$ be, respectively, the set of all positive and negative eigenvalues of T_t, arranged as

$$\lambda_1(t) \geqslant \lambda_2(t) \geqslant \cdots$$

and

$$\mu_1(t) \leqslant \mu_2(t) \leqslant \cdots .$$

Let $\{\phi_n(x, t)\}_{n=1}^{\infty}$ and $\{\psi_n(x, t)\}_{n=1}^{\infty}$ be the corresponding sets of normalized eigenfunctions. Then

(a) For a fixed i, $\lambda_i(t)$ is a monotone nondecreasing and absolutely continuous function of t, $\mu_i(t)$ a monotone nonincreasing and absolutely continuous function of t.

(b) Let $\{t_n\}$ be a sequence of real numbers decreasing to t_0. Let $\{\lambda_i(t_n)\}_{n=1}^{\infty}$ and $\{\phi_i(x, t_n)\}_{n=1}^{\infty}$ be the corresponding sequences of eigenvalues and their corresponding normalized eigenfunctions, respectively. Then the set $\{\phi_i(x, t_n)\}_{n=1}^{\infty}$ is equicontinuous and uniformly bounded on $[0, t_1]$. Hence we can find a subsequence $\{s_n\}$ of $\{t_n\}$ such that as $s_n \to t_0$, $\phi_i(x, s_n)$ converges uniformly with respect to x to $\phi_i(x, t_o)$, whih is a normalized eigenfunction of $\lambda_i(t_0)$.

(c) Also we have

$$\frac{\lambda_i(s_n)-\lambda_i(t_0)}{s_n-t_0}\int_0^{s_n}\phi_i(x,t_0)\cdot\overline{\phi_i(x,s_n)}\cdot dx$$
$$=\frac{\lambda_i(s_n)}{s_n-t_0}\int_{t_0}^{s_n}\phi_i(x,t_0)\cdot\overline{\phi_i(x,s_n)}\cdot dx$$

[$\{s_n\}$ is the subsequence mentioned in (b)].

Taking the limit as $s_n \searrow t_0$ we have

$$\frac{d}{dt}\lambda_i(t)=\lambda_i(t)\cdot|\phi_i(t,t)|^2 \qquad \text{at } t=t_0. \tag{2.3}$$

If $\lambda_i(t)$ is of multiplicity m then there exist m orthonormal eigenfunctions $\{\phi_{i,j}(x,t)\}_{j=1}^m$ satisfying (2.3). [Similar statements hold for the case $\{t_n\} \nearrow t_0$.]

Results similar to (b) and (c) hold for $\mu_i(t)$ also. More details of these results can be seen in [4]. Results (a)–(c) have been extended [9] for general symmetric kernels on general spaces.

(d) If for a fixed i, $\lambda_i(t)$ is simple for all t belonging to an open interval I, then for each t, we can choose a normalized eigenfunction $\phi_i(x,t)$ corresponding to $\lambda_i(t)$ in such a way that $\phi_i(x,t)$ is continuous on I as a function of t. The continuity is also uniform in x. Similar result holds for $\mu_i(t)$ also [10].

(e) Let T be a normal operator, i.e., $TT^* = T^*T$, where T^* denotes the adjoint of T. Let λ and $\phi(x)$ be an eigenvalue and a corresponding eigenfunction of T, respectively, i.e., $T\phi(x)=\lambda.\phi(x)$. Then it is well known [11] that

$$T^*\phi(x)=\bar{\lambda}\cdot\phi(x),$$
$$TT^*\phi(x)=|\lambda|^2\cdot\phi(x),$$

and if $\{\lambda_n\}_{n=1}^\infty$ is the set of all eigenvalues of T^*, then $\{|\lambda_n|^2\}_{n=1}^\infty$ is the set of all eigenvalues of the symmetric operator TT^*.

(*f*) If $\{\psi_n(x)\}_{n=1}^\infty$ is a given complete orthonormal set of eigenfunctions of TT^* [such a set always exists, since TT^* is symmetric], then it is possible to generate another complete orthonormal set $\{\phi_n(x)\}_{n=1}^\infty$ of eigenfunctions of TT^*, which is also a complete orthonormal set of eigenfunctions of T. In other words, orthonormal eigenfunctions $\phi_i(x)$ corresponding to $|\lambda_i|^2$ of TT^* can be chosen to be also orthonormal eigenfunctions of T corresponding to λ_i [11].

Results (a)–(f) will be extensively used in Section 4 while proving the results of this paper.

In the case of symmetric operators the eigenvalues $\{\lambda_n(t)\}$ are real valued, whereas they could be complex in the case of normal operators. This fact makes it difficult to prove the continuity and differentiability properties of $\lambda_i(t)$ as a function of t. In fact, $\lambda_i(t)$ may not be a continuous function, in general, in the case of normal operators. But we can define a new function $\lambda^i(t)$ which takes its value for each t from the spectrum $S_t = \{\lambda_n(t)\}_{n=1}^{\infty}$ of the normal operator T_t and prove that $\lambda^i(t)$ is a continuous function of t and it is differentiable almost everywhere. We can also prove the formula (2.3) for $\lambda^i(t)$. A precise statement of these is given in the theorems below.

From a geometrical view point, the eigenvalues of symmetric integral operators can be plotted on a two-dimensional $(t, \lambda(t))$ plane and for each t, the spectrum of $T_t - \{\lambda_n(t), \mu_m(t)\}_{n,m=1}^{\infty}$ is scattered on a straight line parallel to $\lambda(t)$ axis. Then for a fixed i, graphically we can see $\lambda_i(t)$ and $\mu_i(t)$ to be continuous and almost everywhere differentiable functions of t. But in the case of normal operators, since the spectrum consists of complex eigenvalues, the situation has to be considered in a 3-dimensional space. Hence argument of the complex eigenvalues plays an important role in the continuity and differentiability properties of the eigenvalues of normal operators. Throughout this paper the spectrum $\{\lambda_n(t)\}_{n=1}^{\infty}$ of T_t is arranged such that

$$|\lambda_1(t)| \geqslant |\lambda_2(t)| \geqslant \cdots \geqslant |\lambda_i(t)| \geqslant \cdots, \tag{2.4}$$

for each t. We assume the following conditions on the kernel $K(x)$ throughout.

(i) $K(x)$ is locally square integrable on R^1 (2.5)

(ii) The integral operator T_t generated by $K(x)$ is normal, i.e., $T_t T_t^* = T_t^* T_t$ for each $t \in [0, \infty)$. (2.6)

Since T_t is normal and completely continuous, from [11] it follows that

$$\int_0^t \int_0^t |K(x-y)|^2 \cdot dx \cdot dy = \sum_{n=1}^{\infty} |\lambda_n(t)|^2.$$

Then, following the proof of Vittal Rao [4, Theorem 2], it can be easily seen that $|\lambda_i(t)|$ is a continuous function of t. But since $\arg \lambda_i(t)$ is not necessarily continuous in general, $\lambda_i(t)$ need not be continuous. Hereafter we denote $\arg \lambda(t)$ by $A\lambda_i(t)$ for simplicity.

Following are the main results of this paper, proofs of which will be given in Section 4.

THEOREM 2.1. *If at $t=t_0$, the eigenvalues $\{\lambda_n(t_0)\}_{n=1}^{\infty}$ of T_{t_0} are such that*

$$|\lambda_1(t_0)| > |\lambda_2(t_0)| > \cdots > |\lambda_i(t_0)| > \cdots; \tag{2.7}$$

then for a fixed i, $\lambda_i(t)$ is continuous and almost everywhere differentiable as a function of t on an open interval I containing t_0.

But instead of the inequality (2.7), if two or more eigenvalues of T_{t_0} have same absolute value, i.e., if

$$|\lambda_i(t_0)| = |\lambda_{i+1}(t_0)| = \cdots = |\lambda_{i+m}(t_0)|$$

for some i, then $\lambda_{i+k}(t)$, $k=0, 1, 2,..., m$, is not continuous at $t=t_0$ in general. In this case we define a function $\lambda^i(t)$ on an interval I containing t_0, through the following theorems in which we prove that $\lambda^i(t)$ is continuous, almost everywhere differentiable and satisfies the derivative formula (2.3) on I. Finally we extend the function throughout a sufficiently large interval $(0, M)$ on which the extended function is continuous, almost everywhere differentiable and satisfies the formula (2.3).

THEOREM 2.2. *If at $t=t_0$ and for a fixed i,*

$$|\lambda_{i-1}(t_0)| > |\lambda_i(t_0)| = |\lambda_{i+1}(t_0)| > |\lambda_{i+2}(t_0)| \tag{2.8}$$

and

$$A\lambda_i(t_0) > A\lambda_{i+1}(t_0), \tag{2.9}$$

then there exists an open interval $I=(a_1, a_2)$ containing t_0 such that $I=I_1 \cup I_2$, where

$$I_1 = \{t \in I / A\lambda_i(t) > A\lambda_{i+1}(t)\},$$
$$I_2 = \{t \in I / A\lambda_i(t) < A\lambda_{i+1}(t)\},$$

and at the end points a_1 and a_2 of the open interval I one of the following holds

$$|\lambda_i(t)| > |\lambda_{i+1}(t)| \quad \textit{and} \quad A\lambda_i(t) = A\lambda_{i+1}(t), \tag{A}$$

$$|\lambda_{i-1}(t)| = |\lambda_i(t)| \quad \textit{and/or} \quad |\lambda_{i+1}(t)| = |\lambda_{i+2}(t)|, \tag{B}$$

$$\lambda_i(t) = \lambda_{i+1}(t), \tag{C}$$

$$\text{(A)} \quad \textit{and} \quad \text{(B)},$$

or (D)

$$\text{(B)} \quad \textit{and} \quad \text{(C)}.$$

THEOREM 2.3. *If, at $t=t_0$ and for a fixed i, conditions* (2.8) *and* (2.9) *are satisfied, then on the open interval I defined in the above theorem, we define two functions $\lambda^i(t)$ and $\lambda^{i+1}(t)$ as follows:*

$$\lambda^i(t)=\begin{cases}\lambda_i(t), & \forall t\in I_1,\\ \lambda_{i+1}(t), & \forall t\in I_2,\end{cases}$$

$$\lambda^{i+1}(t)=\begin{cases}\lambda_{i+1}(t), & \forall t\in I_1,\\ \lambda_i(t), & \forall t\in I_2.\end{cases}$$

Then $\lambda^i(t)$ and $\lambda^{i+1}(t)$ are continuous at $t=t_0$ in the sense that $Lt_{t\to t_0}\lambda^i(t)=\lambda^i(t_0)=\lambda_i(t_0)$ and $Lt_{t\to t_0}\lambda^{i+1}(t)=\lambda^{i+1}(t_0)=\lambda_{i+1}(t_0)$.

THEOREM 2.4. *The functions $\lambda^i(t)$ and $\lambda^{i+1}(t)$ are continuous and almost everywhere differentiable on $I=(a_1, a_2)$. Furthermore there exist normalized eigenfunctions $\phi^l(x, t)$ corresponding to $\lambda^l(t)$, $l=1, 2$, such that*

$$\frac{d}{dt}\lambda^l(t)=\lambda^l(t)\cdot|\phi^l(t, t)|^2, \qquad l=i, i+1. \tag{2.10}$$

THEOREM 2.5. *If* (A) *and/or* (B) *of Theorem* 2.2 *hold at a_1 and a_2 , then the functions $\lambda^i(t)$ and $\lambda^{i+1}(t)$ can be extended outside the interval I, preserving the properties mentioned in Theorem* 2.4.

In the above theorems we considered the simple case when two eigenvalues $\lambda_i(t)$ and $\lambda_{i+1}(t)$ have same absolute value. Extension of these results for general cases follows easily from the proofs of Section 4. The extension of $\lambda^i(t)$ and $\lambda^{i+1}(t)$ outside when (C) and/or (D) hold at a_1 and a_2 is also described in Section 4.

Before stating the next theorem, which is for general finite section integral operators, we introduce some notation, which are the same as used by Athreya and Vittal Rao [9].

Let V be a topological space; (V, B, μ) a complete measure space with the σ-algebra big enough to include all Borel sets. Let $\{V_t: t\geq 0\}$ be a family of B-measurable subsets of V such that,

(i) $\mu(V_0)=0$ and $V_t\subset V_s$ whenever $t\leqslant s$, (2.11)

(ii) V_t is compact for each t, (2.12)

(iii) the map $t\to m(t)=\mu(V_t)$ is continuous as a map from $[0, \infty)$ to $[0, \infty)$. (2.13)

Let $K(x, y)$ be a B-measurable complex-valued function on $V\times V$, such that,

(i) $K\in L_2(V_t\times V_t)$, for every $t\geqslant 0$. (2.14)

(ii) The integral operator T_t on $L_2(V_t)$ generated by the kernel $K(x, y)$ defined by $T_t f(x) = \int_{V_t} K(x, y) \cdot f(y) \cdot \mu(dy)$ is a Normal operator, (2.15)

(iii) For every x in V_t, the function $K_x(s)$ defined as $K_x(s) = K(s, x)$ is in $L_2(V_t)$, and (2.16)

(iv) the map $x \to K(x, \cdot)$ is continuous as a map from V_t to $L_2(V_t)$. (2.17)

We further assume that the measure μ admits a polar representation with respect to the family $\{V_t\}$ in the following sense:

There exists a family of $\mathcal{B}$-measurable sets $\{P_t\}$ and a family of probability measures $\{\mu_t\}$ on $\{P_t\}$ such that for all real continuous functions f on V_t,

$$\int_{V_t} f \cdot dm = \int_{[0,t]} \left[\int_{P_t} f(x) \cdot d\mu(x) \cdot \right] dm(\omega) \tag{2.18}$$

and

$$\omega \to \int_{P_\omega} f(x) \cdot d\mu(x) \text{ is continuous}$$

(This is a generalization of the usual spherical polar decomposition with V_t denoting the role of the solid sphere of radius t; P_ω denoting the surface of the sphere of radius ω, μ_ω denoting the normalized surface area measure).

THEOREM 2.6. *With the above notations and assumptions, the results of Theorems* 2.1–2.5 *hold for the operator* T_t *on* $L_2(V_t)$, *with Eq.* (2.10) *taking the form*

$$\frac{d}{dm} \lambda^l(t) = \lambda^l(t) \int_{P_t} |\phi^l(x, t)|^2 \cdot \mu_t(dx), \qquad l = i, i+1 \tag{2.19}$$

almost everywhere with respect to $\alpha(t)$.

Remark. In particular, taking $V = R^1$, $V_t = [0, t]$, with $m(t) = t$, μ_t as the delta measure and $V_t = \{t\}$, we get that Theorems 2.1–2.5 hold for the operator T_t without assuming the kernel to be a difference kernel (i.e., without T_t being a FSWH operator), but just assuming $K(x, y)$ to satisfy (2.14)–(2.17).

3. EIGENVALUES OF SYMMETRIC OPERATOR

In this section we consider T_t, $0 \leqslant t \leqslant M < \infty$, with kernel $K(x)$ satisfying (2.1) and (2.2). For a fixed i, let $\lambda_i(t)$ and $\phi_i(x, t)$ be, respectively, an eigenvalue and its corresponding normalized eigenfunction of T_t. Since we consider the results for a fixed i, we drop the subscript i from $\lambda_i(t)$ and $\phi_i(x, t)$. We define the set $E_1 \subset (0, M)$ as

$$E_1 = \{t \in (0, M)/\lambda(t) \text{ is simple}\}.$$

Then E_1 is open [10] and hence $E_1 = \bigcup_{k=1}^{\infty} I_k^1$, where $I_k^1 = (t_{k,1}^1, t_{k,2}^1)$ are disjoint open intervals. It is evident that $\lambda(t_{k,2}^1)$, $k = 1, 2, \ldots$, have multiplicity > 1. Let

$$B_1 = \{t_{k,1}^1, t_{k,2}^1; k = 1, 2, \ldots\};$$

then B_1 is countable.

Now consider the set $C_1 = (0, M) - \bar{E}_1$ ($\bar{E}_1$ denotes the closure of E_1). It can be easily seen that C_1 is open and $C_1 = \{t \in (0, M) - B_1/\lambda(t)$ has multiplicity $\geqslant 2\}$. If for some $t_0 \in C_1$, $\lambda(t_0)$ has multiplicity two, then there exists an open interval I_0 containing t_0 such that $\lambda(t)$ is of multiplicity two for all $t \in I_0$. For, if not, we can find a sequence $\{t_n\} \subset C_1$ decreasing to t_0 such that $\lambda(t_n)$ are of multiplicity greater than two. Then, corresponding to each $\lambda(t_n)$ we can find at least three orthonormal eigenfunctions $\phi^1(x, t_n)$, $\phi^2(x, t_n)$ and $\phi^3(x, t_n)$. From (*b*) of Section 2 the sequences $\{\phi^l(x, t_n)\}$, $l = 1, 2, 3$, have subsequences $\{\phi^l(x, s_n)\}$, $l = 1, 2, 3$, such that $\operatorname{Lim}_{s_n \to t_0} \phi^l(x, s_n) \to \phi^l(x, t_0)$, where $\phi^l(x, t_0)$, $l = 1, 2, 3$, are three orthonormal eigenfunctions corresponding to $\lambda(t_0)$. This contradicts the fact that $\lambda(t_0)$ is of multiplicity two. Hence we have proved that

$$E_2 = \{t \in C_1/\lambda(t) \text{ has multiplicity } 2\}$$

is open and we can write E_2 as $E_2 = \bigcup_{k=1}^{\infty} I_k^2$, where $I_k^2 = (t_{k,1}^2, t_{k,2}^2)$ are open intervals and $\lambda(t_{k,1}^2)$ and $\lambda(t_{k,2}^2)$, $k = 1, 2, \ldots$, are of multiplicity > 2. The set

$$B_2 = \{t_{k,1}^2, t_{k,2}^2; k = 1, 2, \ldots\}$$

is countable. Proceeding in the same way we get the sets

$$C_n = C_{n-1} - \bar{E}_n,$$
$$B_n = \{t_{k,1}^n, t_{k,2}^n; k = 1, 2, \ldots\}$$

and

$$E_{n+1} = \{t \in C_n/\lambda(t) \text{ has multiplicity } n+1\}$$

and prove that C_n, E_{n+1} are open and B_n is countable. Now we consider the set $B = \bigcup_{r=1}^{\infty} B_r$. B consists of all the boundary points of the open intervals I_k^r for all k and r. It can be easily seen that B is countable. Now let us prove that B is closed in $(0, M)$. Let t_0 be a limit point of B and $\lambda(t_0)$ be of multiplicity m. If $t_0 \notin B$, then $t_0 \in E_m$. E_m being open there exists an open interval $(t_0 - h, t_0 + h) \subset E_m$. Since $\bigcup_k E_k$ and B are disjoint by construction it follows that $(t_0 - h, t_0 + h) \cap B$ is empty, contradicting that t_0 is a limit point of B. Hence $t_0 \in B$ and hence B is closed.

Finally we note that a point t such that $\lambda(t)$ has multiplicity m; $m = 1, 2, \ldots$, is a limit point of B iff there exist at least two sequences $\{t_n\}$ and $\{s_n\}$ such that $\lambda(t_n)$ and $\lambda(s_n)$ have multiplicity m_1 and m_2, respectively, such that $m_1, m_2 \leqslant m$ and $m_1 \neq m_2$. For, if there exist no such sequences then $t \in E_m$ and hence $t \notin B$. Hence t cannot be a limit point of B, since B is closed. Thus we have proved the following theorem.

THEOREM 3.1. *Consider the operator T_t defined by* (1.1) *for* $0 \leqslant t \leqslant M < \infty$, *with kernel $K(x)$ satisfying the conditions* (2.1) *and* (2.2). *With the same notations as above let $E = (0, M) - B$. Then E is open and the sets*

$$E_m = \{t \in E/\lambda(t) \text{ has multiplicity } m\} \quad m = 1, 2, \ldots,$$

are open in E. If

$$\begin{aligned} B_0 = \{t \in (0, M)/&\text{there exist sequences } \{t_n\} \to t \text{ and} \\ &\{s_n\} \to t \text{ such that } \lambda(t_n) \text{ has multiplicity } p \\ &\text{and } \lambda(s_n) \text{ has multiplicity } q, p \neq q\}. \end{aligned}$$

Then B_0 is the set of all limit points of B and $B_0 \subset B$, and hence it is countable.

Though we considered FSWH symmetric operator T_t in the above theorem, it is valid for symmetric operators with general kernels also, because the results on the FSWH symmetric operators which have been used in the proof are true for general symmetric integral operators also [9].

4. Proofs of the Results in Section 2

First, we prove the following two lemmas. Throughout the section we assume the conditions (2.5) and (2.6) on the kernel $K(x)$.

LEMMA 4.1. *For i fixed, if at $t = t_0$*

$$|\lambda_{i-1}(t_0)| > |\lambda_i(t_0)| > |\lambda_{i+1}(t_0)|, \tag{4.1}$$

then $\lambda_i(t)$ is continuous at $t = t_0$.

LEMMA 4.2. *Under the condition* (4.1), *there exists an open interval* $I=(a_1, a_2)$ *containing* t_0 *such that* $\lambda_i(t)$ *is continuous and almost everywhere differentiable on* I. *At* a_1 *and* a_2 *at least one of the inequalities in* (4.1) *becomes an equality.*

Proof of Theorem 2.1. Follows from the above two lemmas.

Proof of Lemma 4.1. The operator T_t is normal and completely continuous and hence its spectrum S_t consists only of at most countable number of eigenvalues $\{\lambda_n(t)\}_{n=1}^{\infty}$. From (e) of Section 2, $\{|\lambda_n(t)|^2\}_{n=1}^{\infty}$ is the set of all eigenvalues of the symmetric, positive operator $T_t^* T_t$. Also from (f) of Section 2 we can choose a function $\phi_i(x, t)$ as a normalized eigenfunction of both $T_t^* T_t$ and T_t corresponding to the eigenvalues $|\lambda_i(t)|^2$ and $\lambda_i(t)$, respectively.

Now we consider a sequence $\{t_n\}$ decreasing to t_0 and the corresponding sequences of eigenvalues $\{\lambda_i(t_n)\}$ and their corresponding eigenfunctions $\{\phi_i(x, t_n)\}$. Since $\phi_i(x, t_n)$ is also an eigenfunction of $T_t^* T_t$ corresponding to $|\lambda_i(t_n)|^2$, from (b) of Section 2, we can choose a subsequence $\{s_n\}$ of $\{t_n\}$ such that $\phi_i(x, s_n) \to \phi_i(x, t_0)$ as $s_n \to t_0$ where $\phi_i(x, t_0)$ is a normalized eigenfunction of $T_t^* T_t$ corresponding to $|\lambda_i(t_0)|^2$. It can be easily seen that

$$\lambda_i(t) = \int_0^t \left(\int_0^t K(x-y) \cdot \phi_i(y, t)\, d\, dy \right) \cdot \overline{\phi_i(x, t)} \cdot dx.$$

Now for $s_m < s_n$; $|\lambda_i(s_n) - \lambda_i(s_m)|$ is equal to

$$\left| \int_0^{s_m} \int_0^{s_m} K(x-y) \cdot [\phi_i(y, s_n)\, \overline{\phi_i(x, s_n)} - \phi_i(y, s_m)\, \overline{\phi_i(x, s_m)}]\, dy \cdot dx \right.$$
$$\left. + \left(\int_{s_m}^{s_n} \int_0^{s_n} + \int_0^{s_m} \int_{s_m}^{s_n} \right) (K(x-y)\, \phi_i(y, s_n)\, \overline{\phi_i(x, s_n)}\, dy \cdot dx) \right|$$
$$\leqslant \left| \int_0^{s_m} \int_0^{s_m} K(x-y)\, [\phi_i(y, s_n)\, \overline{\phi_i(x, s_n)} - \phi_i(y, s_m)\, \overline{\phi_i(x, s_m)}]\, dx \cdot dy \right|$$
$$+ \left| \left(\int_{s_m}^{s_n} \int_0^{s_n} + \int_0^{s_m} \int_{s_m}^{s_n} \right) (K(x-y)\, \phi_i(y, s_n) \cdot \overline{\phi_i(x, s_n)} \cdot dy \cdot dx \right|.$$

The first term converges to zero as n and m tend to ∞, since $\{\phi_i(x, s_n)\}_{n=1}^{\infty}$ is equicontinuous and uniformly bounded, (from (b) of Sect. 2). The second term tends to zero because $\{\phi_i(x, s_n)\}_{n=1}^{\alpha}$ is uniformly bounded and $K(x)$ is locally square integrable. Hence $\{\lambda_i(s_n)\}$ is a Cauchy sequence and hence converges to $\lambda(t_0)$ (say) as $s_n \to t_0$. We know that

$$\lambda_i(s_n)\, \phi_i(x, s_n) = \int_0^{s_n} K(x-y)\, \phi_i(y, s_n)\, dy \qquad \text{for all } n.$$

Taking the limit as $s_n \to t_0$ and using that ϕ's converge uniformly with respect to x we get

$$\lambda(t_0)\,\phi_i(x, t_0) = \int_0^{t_0} K(x-y)\,\phi_i(y, t_0)\,dy. \tag{4.3}$$

From (4.3) we infer that $\lambda(t_0)$ is an eigenvalue of T_{t_0} with corresponding eigenfunction $\phi_i(x, t_0)$. But $\phi_i(x, t_0)$ is an eigenfunction of $T_{t_0}^* T_{t_0}$ corresponding to $|\lambda_i(t_0)|^2$. Now operating $T_{t_0}^*$ on both sides of (4.3) we get $|\lambda(t_0)|^2 = |\lambda_i(t_0)|^2$. But from (4.1) it follows that there exists only one eigenvalue $\lambda_i(t_0)$ with absolute value $|\lambda_i(t_0)|$. Hence $\lambda(t_0) = \lambda_i(t_0)$. Thus any arbitrary sequence $\{t_n\}$ decreasing to t_0 has a subsequence $\{s_n\}$ such that $\lambda_i(s_n) \to \lambda_i(t_0)$ as $s_n \to t_0$. We can argue in a similar way when $\{t_n\}$ increases to t_0. Hence $\lambda_i(t)$ is continuous at $t = t_0$.

Note. If $\{t_n\} \to t_0$, in the following, whenever there exists a subsequence $\{s_n\}$ of $\{t_n\}$ such that $\lambda_i(s_n) \to \lambda_i(t_0)$ we denote the subsequence also by $\{t_n\}$ for simplicity.

Proof of Lemma 4.2. We assert that there exists an open interval I containing t_0 such that the hypothesis (4.1) holds for all $t \in I$. If not, we can find a sequence $\{t_n\}$ converging to t_0 such that at least two of the eigenvalues (among λ_{i-1}, λ_i and λ_{i+1}) have the same absolute values. Without loss of generality we assume that $|\lambda_{i-1}(t_n)| = |\lambda_i(t_n)|$, $\forall n$. Now applying (b) of Section 2 to the symmetric operator $T_t^* T_t$ we can show that

$$|\lambda_{i-1}(t_n)| \to |\lambda_{i-1}(t_0)| \qquad \text{and} \qquad |\lambda_i(t_n)| \to |\lambda_i(t_0)|,$$

which imply that $|\lambda_{i-1}(t_0)| = |\lambda_i(t_0)|$, which is a contradiction.

Let $I = (a_1, a_2)$ be the maximal interval in which the hypothesis (4.1) holds. At a_1 and a_2 (4.1) obviously does not hold, since otherwise using the above arguments we can extend I outside a_1 and a_2 which contradicts the maximal property of I. Now since (4.1) holds for all $t \in I$, the continuity of $\lambda_i(t)$ at each point of I follows from Lemma 4.1. Now we prove the differentiability property of $\lambda_i(t)$ and the formula (2.3) on I.

Since the operator $T_t^* T_t$ is symmetric for all t with eigenvalues $\{|\lambda_n(t)|^2\}_{n=1}^{\infty}$, from (b) and (c) of Section 2, it follows that there exists a sequence $\{t_n\} \subset I$ decreasing to $t \in I$ such that

$$\frac{|\lambda_i(t_n)|^2 - |\lambda_i(a)|^2}{t_n - a} \int_{t_n}^{a} \phi_i(x, t_n)\,\phi_i(x, a), \tag{4.4}$$

for a suitable eigenfunction $\phi_i(x, a)$. We get a similar equation for $\{t_n\}$ increasing to a. As $t_n \to a$ we get

$$\frac{d}{dt}\,|\lambda_i(t)|^2 = |\lambda_i(t)|^2\,|\phi_i(t, t)|^2 \qquad \text{at } t = a,$$

whenever $|\lambda_i(t)|^2$ is differentiable at $t=a$. [It should be noted that $|\lambda_i(t)|^2$ is differentiable almost everywhere on $[0, \infty)$—from (a)—Sect. 2.] Hence for a suitable choice of $\phi_i(x, a)$ we get

$$\operatorname*{Lt}_{t_n \to a} \frac{1}{t_n - a} \int_{t_n}^{a} \phi_i(x, t_n)\, \overline{\phi_i(x, a)}\, dx = \frac{d/dt\, |\lambda_i(a)|^2}{|\lambda_i(a)|^2}. \tag{4.5}$$

Now consider the sequence $\{\lambda_i(t_n)\}$ of eigenvalues corresponding to $\{t_n\}$. From (f) of Section 2 it is possible to choose $\phi_i(x, t_n)$ for each t_n to be an eigenfunction corresponding to both $|\lambda_i(t_n)|^2$ of $T_{t_n}^* T_{t_n}$ and $\lambda_i(t_n)$ of T_{t_n} simultaneously. Moreover, since T_{t_n} is normal, we have

$$\begin{aligned}
&\frac{\lambda_i(t_n) - \lambda_i(a)}{t_n - a} \int_0^{t_n} \phi_i(x, t_n)\, \overline{\phi_i(x, a)}\, dx \\
&\quad = \frac{1}{t_n - a} \left[\lambda_i(t_n) \int_0^{t_n} \phi_i(x, t_n)\, \overline{\phi_i(x, a)}\, dx \right. \\
&\qquad \left. - \int_0^{t_n} \phi_i(x, t_n) \left[\int_0^a K(x-y)\, \overline{\phi_i(y, a)}\, dy \right] dx \right] \\
&\quad = \frac{1}{t_n - a} \lambda_i(t_n) \int_{t_n}^{a} \phi_i(x, t_n)\, \overline{\phi_i(x, a)}\, dx.
\end{aligned} \tag{4.6}$$

We know that $\operatorname{Lim}_{t_n \to a} \lambda_i(t_n) = \lambda_i(a)$ and $\operatorname{Lim}_{t_n \to a} \int_0^{t_n} \phi_i(x, t_n) \times \overline{\phi_i(x, a)}\, dx = 1$. Taking limits on both sides of (4.6) we get

$$\begin{aligned}
\operatorname*{Lim}_{t_n \to a} \frac{\lambda_i(t_n) - \lambda_i(a)}{t_n - a} .1 &= \lambda_i(a) \\
&\quad \times \operatorname*{Lim}_{t_n \to a} \frac{\int_{t_n}^{a} \phi_i(x, t_n)\, \overline{\phi_i(x, a)}\, dx}{t_n - a}.
\end{aligned} \tag{4.7}$$

From (4.5) it is evident that the value of the limit of the r.h.s. of (4.7) is independent of the choice of the sequence $\{t_n\}$. Hence the l.h.s. of (4.7) exists for any arbitrary choice of the sequence $\{t_n\}$ [the limits exists only for a subsequence of $\{t_n\}$ though we do not mention it every time] and hence (4.7) can be written as

$$\frac{d}{dt} \lambda_i(t) = \lambda_i(t)\, |\phi_i(t, t)|^2 \qquad \text{at } t = a, \tag{4.8}$$

for a suitable choice of $\phi_i(x, t)$. The limit (4.5) exists only when $|\lambda_i(t)|^2$ is differentiable at $t=a$. Hence (4.8) holds only when $|\lambda_i(t)|^2$ is differentiable at $t=a$.

Proof of Theorem 2.2. From the proof of Lemma 4.1 we know that there exists a sequence $\{t_n\}$ converging to t_0 such that $\lambda_i(t_n) \to \lambda_i(t_0)$ where $\lambda(t_0)$ satisfies the condition $|\lambda(t_0| = |\lambda_i(t_0)|$. Similarly we have $\lambda_{i+1}(t_n) \to \mu(t_0)$ where $|\mu(t_0)| = |\lambda_{i+1}(t_0)|$. But since $|\lambda_i(t_0)| = |\lambda_{i+1}(t_0)|$, $\lambda(t_0)$ could be any one of $\lambda_i(t_0)$ and $\lambda_{i+1}(t_0)$. Similarly $\mu(t_0)$ could be one of $\lambda_{i+1}(t_0)$ and $\lambda_i(t_0)$, i.e., the limits of the sequences $\{\lambda_i(t_n)\}$ and $\{\lambda_{i+1}(t_n)\}$ depend on the choice of the sequence $\{t_n\}$. Hence we can find sequences $\{s_n\}$ and $\{r_n\}$ decreasing to t_0 such that

$$\lim_{s_n \to t_0} \lambda_i(s_n) = \lambda_i(t_0) \qquad \text{and} \qquad \lim_{r_n \to t_0} \lambda_i(r_n) = \lambda_{i+t}(t_0)$$

and

$$\lim_{s_n \to t_0} \lambda_{i+1}(s_n) = \lambda_{i+1}(t_0) \qquad \text{and} \qquad \lim_{r_n \to t_0} \lambda_{i+1}(r_n) = \lambda_i(t_0).$$

Now if there exists a sequence $\{a_n\} \to t_0$ such that $A\lambda_i(a_n) = A\lambda_i(a_n)$, $\forall n$, then there exists a subsequence $\{b_n\}$ of $\{a_n\}$ such that $\lambda_i(b_n) \to \lambda_i(t_0)$ (or to $\lambda_{i+1}(t_0)$) and $\lambda_{i+1}(b_n) \to \lambda_{i+1}(t_0)$ (or to $\lambda_i(t_0)$) which imply that $A\lambda_i(t_0) = A\lambda_{i+1}(t_0)$, a contradiction to (2.9). Hence there exists an interval I containing t_0 such that $A\lambda_i(t) \neq A\lambda_{i+1}(t)$, $t \in I$. Thus the first part of the theorem is proved.

(b) Since

$$|\lambda_{i-1}(t_0)| > |\lambda_i(t_0)| \tag{4.9}$$

and

$$|\lambda_{i+1}(t_0)| > |\lambda_{i+2}(t_0)|, \tag{4.10}$$

we can find open intervals L_1 and L_2 containing t_0 such that (4.9) holds on L_1 and (4.10) holds on L_2. If $L_1 \cap L_2 = (t_1, t_2)$, then both (4.9) and (4.10) hold on L. We assume that L is maximal such interval. Then at t_1 and t_2, at least one of the conditions (4.9) and (4.10) is violated, i.e., the strict inequality becomes equality.

Now consider the interval $I = (a_1, a_2)$ of part (a) which is maximal. If a_1 is an interior point of L then we have at a_1, $|\lambda_{i-1}(a_1)| > |\lambda_i(a_1)| \geqslant \lambda_{i+1}(a_1)| > \lambda_{i+2}(a_1)$. If $A\lambda_i(a_1) \neq A\lambda_{i+1}(a_1)$, then we can extend I outside a_1 (using Lemma 4.1 if $|\lambda_i(a_1)| > |\lambda_{i+1}(a_1)|$, otherwise using part (a)) which is contrary to the maximal property. Hence $A\lambda_i(a_1) = A\lambda_{i+1}(a_1)$. Similar argument hold for a_2 also.

If a_1 and a_2 are not interior points of L, then $I = L$ and hence all the other alternatives mentioned in (b) are possible at a_1 and a_2.

Proof of Theorem 2.3. The assumptions at $t = t_0$ are

$$|\lambda_{i-1}(t_0)| > \lambda_i(t_0) = |\lambda_{i+1}(t_0)| > |\lambda_{i+2}(t_0)| \tag{2.8}$$

and

$$A\lambda_i(t_0) > A\lambda_{i+1}(t_0). \tag{2.9}$$

As shown in Theorem 2.2 we can find a sequence $\{t_n\}$ converging to t_0 such that $\lambda_i(t_n)$ converges either to $\lambda_i(t_0)$ or to $\lambda_{i+1}(t_0)$ and $\lambda_{i+1}(t_0)$ converges either to $\lambda_{i+1}(t_0)$ or to $\lambda_i(t_0)$. Now consider the open interval I mentioned in Theorem 2.2. The following cases arise:

Case (i). $|\lambda_i(t)| = |\lambda_{i+1}(t)|$ and $A\lambda_i(t) > A\lambda_{i+1}(t)$, $\forall t \in I$.

Case (ii). $|\lambda_i(t)| > |\lambda_{i+1}(t)|$ and $A\lambda_i(t) > A\lambda_{i+1}(t)$, $\forall t \in I - t_0$

Case (iii). $|\lambda_i(t)| > |\lambda_{i+t}(t)|$ and $A\lambda_i(t) > A\lambda_{i+1}(t)$, $\forall t \in I$ and $t < t_0$ and

$$|\lambda_i(t)| = |\lambda_{i+1}(t)| \quad \text{and} \quad A\lambda_i(t) > A\lambda_{i+1}(t), \quad \forall t \in I \text{ and } t \geqslant t_0$$

or vice versa.

Case (iv). $|\lambda_i(t)| > |\lambda_{i+1}(t)|$ and $A\lambda_1(t) > A\lambda_{i+1}(t)$, $\forall t \in I$ and $t < t_0$ and

$$|\lambda_i(t)| > |\lambda_{i+1}(t)| \quad \text{and} \quad A\lambda_{i+1}(t), \quad \forall t \in I \text{ and } t > t_0$$

or vice versa

Case (v). We can find sequences $\{t_n\}$, $\{s_n\}$, and $\{r_n\}$ in I decreasing or increasing to t_0 such that

$$|\lambda_i(t_n)| = |\lambda_{i+1}(t_n)| \quad \text{and} \quad A\lambda_i(t_n) > A\lambda_{i+1}(t_n), \tag{4.11}$$

$$|\lambda_i(s_n)| > |\lambda_{i+1}(s_n)| \quad \text{and} \quad A\lambda_i(s_n) > A\lambda_{i+1}(s_n), \tag{4.12}$$

and

$$|\lambda_i(r_n)| > |\lambda_{i+1}(r_n)| \quad \text{and} \quad A\lambda_1(r_n) < A\lambda_{i+1}(r_n). \tag{4.13}$$

Note 1. If we apply Theorem 3.1 for the symmetric operator $T_t^* T_t$ we can see that the set of all points like to defined in case (v) is countable.

Note 2. If for any $t \in I$, the ith and $(i+1)$th eigenvalues have the same absolute value, then the one with the greater argument is renamed as $\lambda_i(t)$ and the other one $\lambda_{i+1}(t)$ as in (2.9).

Now we give the proof of the continuity of $\lambda_i(t)$ at t_0.

In cases (i)–(iii), any sequence $\{t_n\}$ decreasing (increasing) to t_0 has a subsequence $\{s_n\}$ such that $\lambda_i(s_n) \to \lambda_i(t_0)$ and $\lambda_{i+1}(s_n) \to \lambda_{i+1}(t_0)$, since $A\lambda_i(t_n) > A\lambda_{i+1}(t_n)$, $\forall n$. Hence $\lambda_i(t)$ and $\lambda_{i+1}(t)$ are continuous at $t = t_0$.

Case (iv). If $\{t_n\}$ any arbitrary sequence in I increasing to t_0 then

$$\lambda_i(t_n) \to \lambda_i(t_0) \quad \text{and} \quad \lambda_{i+1}(t_n) \to \lambda_{i+1}(t_0). \tag{4.14}$$

If $\{s_n\}$ is any arbitrary sequence in I decreasing to t_0 then

$$\lambda_i(s_n) \to \lambda_{i+1}(t_0) \quad \text{and} \quad \lambda_{i+1}(s_n) \to \lambda_i(t_0). \tag{4.15}$$

We get (4.14) and (4.15) using the argument conditions given in Case (iv). From (4.14) and (4.15) it is evident that $\lambda_i(t)$ and $\lambda_{i+1}(t)$ are not continuous at t_0. Now let us define the function $\lambda^i(t)$ and $\lambda^{i+1}(t)$ on I as follows:

$$\lambda^i(t) = \begin{cases} \lambda_i(t), & t \in I \text{ and } t \leqslant t_0, \\ \lambda_{i+1}(t), & t \in I \text{ and } t > t_0, \end{cases}$$

$$\lambda^{i+1}(t) = \begin{cases} \lambda_{i+1}(t), & \forall t \in I \text{ and } t \leqslant t_0, \\ \lambda_i(t), & \forall t \in I \text{ and } t > t_0. \end{cases}$$

From (4.14) and (4.15) it can be easily seen that $\lambda^i(t)$ and $\lambda^{i+1}(t)$ are continuous at t_0.

Case (v). Let us consider the following subsets of I.

$$E = \{t \in I / |\lambda_i(t)| = |\lambda_{i+1}(t)| \text{ and } A\lambda_i(t) > A\lambda_{i+1}(t)\},$$

$$E_1 = \{t \in I / |\lambda_i(t)| > |\lambda_{i+1}(t)| \text{ and } A\lambda_i(t) > A\lambda_{i+1}(t)\},$$

$$E_2 = \{t \in I / |\lambda_i(t)| > |\lambda_{i+1}(t)| \text{ and } A\lambda_i(t)| < A\lambda_{i+1}(t)\}.$$

The above sets are disjoint and $I = E \cup E_1 \cup E_2$ (from Theorem 2.2). Now the functions $\lambda^i(t)$ and $\lambda^{i+1}(t)$ are defined as

$$\lambda^i(t) = \begin{cases} \lambda_i(t) & \text{if } t \in E \cup E_1, \\ \lambda_{i+1}(t) & \text{if } t \in E_2, \end{cases}$$

$$\lambda^{i+1}(t) = \begin{cases} \lambda_{i+1}(t) & \text{if } t \in E \cup E_1, \\ \lambda_i(t) & \text{if } t \in E_2. \end{cases}$$

From the definition of $\lambda^i(t)$ it can be easily seen that any sequence decreasing (increasing) to t_0 has a subsequence $\{t_n\}$ such that $\lambda^i(t_n) \to \lambda_i(t_0) = \lambda^i(t_0)$. Similarly $\lambda^{i+1}(t_n) \to \lambda_{i+1}(t_0) = \lambda^{i+1}(t_0)$. Hence $\lambda^i(t)$ and $\lambda^{i+1}(t)$ are continuous at t_0.

Remark 1. In Cases (i)–(iii) $\lambda^i(t) = \lambda_i(t)$ and $\lambda^{i+1}(t) = \lambda_{i+1}(t)$ $t \in I$.

Remark 2. In Theorems 2.2 and 2.3, instead of conditions (2.8) and (2.9), if we assume that

$$|\lambda_{i-1}(t_0)| > |\lambda_i(t_0)| = |\lambda_{i+1}(t_0)| = |\lambda_{i+2}(t_0)| > |\lambda_{i+3}(t_0)|$$

and $A\lambda_i(t_0) > A\lambda_{i+1}(t_0) > A\lambda_{i+2}(t_0)$ then we can find an open interval $I = (a_1, a_2)$ containing t_0, such that no two of the eigenvalues $\lambda_i(t)$, $\lambda_{i+1}(t)$

and $\lambda_{i+2}(t)$ have the same argument at each $t \in I$. At a_1 and a_2 at least one of the following holds

(i) $|\lambda_{i-1}(t)| = |\lambda_i(t)|$ and/or $|\lambda_{i+2}(t)| = |\lambda_{i+3}(t)|$.

(ii) Arguments of at least two of the eigenvalues $\lambda_i(t)$, $\lambda_{i+1}(t)$, and $\lambda_{i+2}(t)$ are equal.

(iii) At least two of these eigenvalues are equal.

We define the following subsets of I.

$$E_1 = \{t \in I / |\lambda_i(t)| \geqslant |\lambda_{i+1}(t)| \geqslant |\lambda_{i+2}(t)| \\ A\lambda_i(t) > A\lambda_{i+1}(t) > A\lambda_{i+2}(t)\},$$

$$E_2 = \{t \in I / |\lambda_i(t)| \geqslant \lambda_{i+1}(t)| > |\lambda_{i+2}(t)| \\ A\lambda_i(t) > A\lambda_{i+2}(t) > A\lambda_{i+1}(t)\},$$

$$E_3 = \{t \in I / |\lambda_i(t) > |\lambda_{i+1}(t)| \geqslant |\lambda_{i+2}(t) \\ A\lambda_{i+1}(t) > A\lambda_i(t) > A\lambda_{i+2}(t)\},$$

$$E_4 = \{t \in I / |\lambda_i(t)| > |\lambda_{i+1}(t)| \geqslant |\lambda_{i+2}(t) \\ A\lambda_{i+2}(t) > A\lambda_i(t) > A\lambda_{i+1}(t)\},$$

$$E_5 = \{t \in I / |\lambda_i(t)| \geqslant |\lambda_{i+1}(t)| > |\lambda_{i+2}(t) \\ A\lambda_{i+2}(t) > A\lambda_i(t) > A\lambda_{i+1}(t)\},$$

$$E_6 = \{t \in I / |\lambda_i(t) > |\lambda_{i+1}(t)| > |\lambda_{i+2}(t) \\ A\lambda_{i+2}(t) > A\lambda_{i+1}(t) > A\lambda_i(t)\}.$$

Then $I = \bigcup_{k=1}^{6} E_k$. We define the functions $\lambda^i(t)$, $\lambda^{i+1}(t)$, and $\lambda^{i+2}(t)$ as follows:

$$\begin{aligned} \lambda^i(t) = \lambda_i(t) \quad & \text{on } E_1 \cup E_2, \\ \lambda_{i+1}(t) \quad & \text{on } E_3 \cup E_4, \\ \lambda_{i+2}(t) \quad & \text{on } E_5 \cup E_6. \end{aligned}$$

$$\begin{aligned} \lambda^{i+\iota}(t) = \lambda_{i+1}(t) \quad & \text{on } E_1 \cup E_6, \\ \lambda_{i+2}(t) \quad & \text{on } E_2 \cup E_4, \\ \lambda_i(t) \quad & \text{on } E_3 \cup E_6. \end{aligned}$$

$$\begin{aligned} \lambda^{i+2}(t) = \lambda_{i+2}(t) \quad & \text{on } E_1 \cup E_3, \\ \lambda_{i+1}(t) \quad & \text{on } E_2 \cup E_5, \\ \lambda_i(t) \quad & \text{on } E_4 \cup E_6. \end{aligned}$$

These function were defined according to the argument conditions at t_0. Now it is easy to show that they are continuous at $t = t_0$, i.e.,

$$\operatorname*{Lim}_{t \to t_0} \lambda^l(t) = \lambda^l(t_0), \qquad l = i, i+1, \text{ and } i+2.$$

If n distinct eigenvalues $\lambda_i(t), \lambda_{i+1}(t),..., \lambda_{i+n-1}(t)$ have same absolute value at $t = t_0$, then as above we can find an open interval I and n subsets of I such that $I = \bigcup_{k=1}^n E_k$ and define n functions $\lambda^i(t), \lambda^{i+1}(t),..., \lambda^{i+n-1}(t)$ in a similar way according to the argument conditions. The continuity of these functions can be proved as in Theorem 2.3.

Remark 3. Since $T_t T_t^*$ is symmetric and completely continuous, the eigenvalues $|\lambda_i(t)|^2$ have finite multiplicity. Hence only a finite number of eigenvalues of T_t can have equal absolute value for each t. Hence the above remark extends Theorems 2.2 and 2.3 for a general situation.

Proof of Theorem 2.4. From the definition of I (of Theorem 2.2) we know that $|\lambda_{i-1}(t)| > |\lambda_i(t)|$ and $|\lambda_{i+1}(t)| > |\lambda_{i+2}(t)|$ $t \in I$. If $t_1 \in E_1$ then at t_1, we have $|\lambda_{i-1}(t_1)| > |\lambda_i(t_1)| > |\lambda_{i+1}(t_1)| > |\lambda_{i+2}(t_1)|$ and

$$A\lambda_i(t_1) > A\lambda_{i+1}(t_1). \tag{4.16}$$

Using Lemma 4.2 it can be easily shown that there exists an open interval I_1 containing t_1, on which (4.16) holds, i.e., $I_1 \subset E_1$. In other words E_1 is open. Hence $E_1 = \bigcup_{n=1}^\infty I_n$, I_n-open interval. Similarly $E_2 = \bigcup_{m=1}^\infty I'_m$, I'_m-open interval. It is evident from Theorem 2.2, that if $I_n = (a_1^n, a_2^n)$, then a_1^n and a_2^n belong to E. [This need not be true when a_1^n and a_2^n are boundary points of I itself, where other alternatives given in theorem 2.2 hold.] Now

$$\begin{aligned} \lambda^i(t) = \lambda_i(t) & \qquad \text{on } \bar{I}_n, n, \\ \lambda_{i+1}(t) & \qquad \text{on } I'_m, m \end{aligned}$$

($\bar{I}_n$ denotes the closure of I_n).

From Lemma 4.2, $\lambda^i(t)$ is continuous on each I_n and I'_n and from Lemma 4.2 $\lambda^i(t)$ is continuous at each point of E. Hence $\lambda^i(t)$ is continuous on I.

Now we prove the differentiability property of $\lambda^i(t)$ on I. Since $\lambda^i(t) = \lambda_i(t)$ on I_n and $\lambda^i(t) = \lambda_{i+1}(t)$ on I'_n for all n, it can be easily seen from Lemma 4.2, that $\lambda^i(t)$ is differentiable almost everywhere on $\bigcup_n I_n$ and $\bigcup_n I'_n$. Moreover it satisfies (2.10) almost everywhere on $E_1 \cup E_2$.

Now let $a \in E$. Then $|\lambda_i(a)| = |\lambda_{i+1}(a)|$. We know from the definitions of $\lambda^i(t)$ that $\lambda^i(t) \to \lambda_i(a)$ as $t \to a$. Since the eigenvalues $\lambda_i(t)$ and $\lambda_{i+1}(t)$ satisfy the hypothesis of Theorem 2.2 at $t = a$, there exists an open interval J containing a as defined in that theorem. Now we consider the five cases considered in Theorem 2.3 on J.

In cases (i), (ii) and (iii), since $\lambda^i(t)=\lambda_i(t)$ on J the formula

$$\frac{d}{dt}\lambda^i(t)=\lambda^i(t)\ |\phi^i(t,t)|^2 \qquad \text{at } t=a$$

can be proved as in lemma 4.2.

Case (iv). For any arbitrary sequence $\{t_n\}\nearrow a$, we have

$$\frac{\lambda^i(t_n)-\lambda^i(a)}{t_n-a}\int_0^{t_n}\phi^i(x,t_n)\cdot\overline{\phi^i(x,a)}\,dx$$

$$=\lambda^i(t_n)\int_{t_n}^{a}\phi^i(x,t_n)\,\overline{\phi^i(x,a)}\cdot dx \tag{4.17}$$

Since $\lambda^i(t)=\lambda_i(t)$ for $t\in J$ and $t\leqslant a$, arguing as in Lemma 4.2 we get for a suitable subsequence $\{s_n\}$ of $\{t_n\}$ such that

$$\operatorname*{Lim}_{s_n\nearrow a}\int_{s_n}^{a}\frac{\phi^i(x,s_n)\,\overline{\phi^i(x,a)}\,dx}{s_n-a}=\frac{d/dt\,|\lambda_i(t)|^2}{|\lambda_i(t)|^2} \qquad \text{at } t=a. \tag{4.18}$$

Similarly for any sequence $\{p_n\}$ a we get

$$\frac{\lambda^i(p_n)-\lambda^i(a)}{p_n-a}\int_0^{p_n}\phi^i(x,p_n)\cdot\phi'(x,a)\cdot dx=\lambda^i(p_n)$$

$$\times\int_a^{p_n}\phi^i(x,p_n)\cdot\overline{\phi^i(x,a)}\cdot dx. \tag{4.19}$$

Since $\lambda^i(t)=\lambda_{i+1}(t)$ for $t\in J$ and $t>a$ we get

$$\operatorname*{Lim}_{r_n\searrow a}\int_a^{r_n}\phi^i(x,r_n)\,\overline{\phi^i(x,a)}\,dx=\frac{d/dt\,|\lambda_{i+1}(t)|^2}{|\lambda_{i+1}(t)|^2} \qquad \text{at } t=a \tag{4.20}$$

for a suitable subsequence $\{r_n\}$ of $\{p_n\}$.

Now viewing the almost everywhere differentiable functions $|\lambda_i(t)|^2$ and $|\lambda_{i+1}(t)|^2$ defined on J geometrically, it can be easily seen that these two curves touch only at the point $t=a$. Hence whenever the derivatives of $|\lambda_{i+1}(t)|^2$ and $|\lambda_i(t)|^2$ exist at $t=a$, they should be equal. Hence the limits given by (4.18) and (4.20) are same. Using this idea in (4.17) and (4.19) and by taking limits we get the formula (2.10).

Case (y). We can find sequences $\{s_n\}$ and $\{r_n\}$ belonging to E_1 and E_2, respectively, such that

$$\int_{s_n}^{a}\phi^i(x,s_n)\cdot\overline{\phi^i(x,a)}\,dx\to\frac{d/dt\,|\lambda_i(t)|^2}{|\lambda_i(a)|^2}, \qquad \text{at} \qquad t=a \tag{4.21}$$

and

$$\int_{r_n}^{a} \phi^i(x, r_n) \cdot \overline{\phi^i(x, a)} \cdot dx \to \frac{d/dt\,|\lambda_{i+1}(t)|^2}{|\lambda_{i+1}(a)|^2} \quad \text{at} \quad t = a. \tag{4.22}$$

But there also exists a sequence $\{t_n\} \nearrow a$ such that $|\lambda_i(t_n)| = |\lambda_{i+1}(t_n)|$ for all n. Applying (c) of Section 2 to the symmetric operator $T_t^* T_t$ we get

$$\frac{d}{dt}\,|\lambda_i(t)|^2 = \frac{d}{dt}\,|\lambda_{i+1}(t)|^2 \qquad \text{at } t = a. \tag{4.23}$$

Now for any arbitrary sequence $\{t_n\} \subset J$ increasing (decreasing) to a we have

$$\frac{\lambda^i(t_n) - \lambda^i(a)}{t_n - a} \int_0^{t_n} \phi^i(x, t_n)\,\overline{\phi^i(x, a)}\,dx = \frac{\lambda^i(t_n)}{t_n - a} \times \int_{t_n}^{a} \phi^i(x, t_n)\,\overline{\phi^i(x, a)}\,dx. \tag{4.24}$$

Thus for a suitable subsequence $\{r_n\}$ of $\{t_n\}$ we have

$$\operatorname*{Lim}_{r_n \nearrow a} \int_0^{r_n} \phi^i(x, r_n)\,\overline{\phi^i(x, a)}\,dx = 1,$$

$$\operatorname*{Lim}_{r_n \nearrow a} \lambda^i(r_n) = \lambda^i(a) = \lambda_i(a),$$

and

$$\operatorname*{Lim}_{r_n \nearrow a} \frac{\int_{r_n}^{a} \phi^i(x, r_n)\,\overline{\phi^i(x, a)}\,dx}{r_n - a} = |\phi^i(a, a)|^2 = \frac{d/dt\,|\lambda_i(a)|^2}{|\lambda_i(a)|^2},$$

a fixed value (from (4.21), (4.22), and (4.23)). Now taking limits on both sides of (4.24) we get (2.10).

Remark 4. Theorem 2.4 can be extended easily to the general case when more than two (distinct) eigenvalues have same absolute value.

Proof of Theorem 2.5. First, we assume that only (A) holds at a_1. Then $|\lambda_{i-1}(a_1)| > |\lambda_i(a_1)| > |\lambda_{i+1}(a_1)| > |\lambda_{i+2}(a_1)|$ and $A\lambda_i(a_1) = A\lambda_{i+1}a_1)$. From Lemmas 4.1 and 4.2, it follows that there exists an open interval I, containing a_1 such that $\lambda_i(t)$ and $\lambda_{i+1}(t)$ are continuous and almost everywhere differentiable on I_1. But from the definition of I and I_1, it can be easily seen that $\lambda^i(t)=\lambda_i(t)$ on $I\cap I_1$. Hence we can extend the function $\lambda^i(t)$ outside I (to the left of a_1) as follows. Denoting the extended function as $\mu^i(t)$ we get

$$\mu^i(t)=\begin{cases}\lambda^i(t) & \text{on } I,\\ \lambda_i(t) & \text{on } I_1.\end{cases}$$

Let $I_1=(a'_1, a'_2)$ and $|\lambda_{i-1}(a'_1)|>|\lambda_i(a'_1)|$ and $|\lambda_{i+1}(a'_1)|>|\lambda_{i+2}(a'_1)|$. Then from Lemma 4.2, $|\lambda_i(a'_1)|=|\lambda_{i+1}(a'_1)|$. Then there exists an open interval I_2 containing a_1 on which we can define a continuous and almost everywhere differentiable function $\lambda^i(t)$ as in Theorem 2.3. It is evident that

$$\gamma^i(t)=\lambda_i(t) \qquad \text{on } I_1\cap I_2.$$

Hence the function $f^i(t)$ defined on $I\cup I_1\cup I_2$ as

$$f^i(t)=\begin{cases}\lambda^i(t) & \text{on } I,\\ \lambda_i(t) & \text{on } I_1,\\ \gamma^i(t) & \text{on } I_2,\end{cases}$$

is an extension of $\lambda^i(t)$ on $I\cup I_1\cup I_2$. This function $f^i(t)$ is continuous-differentiable almost everywhere and satisfies (2.10) on $I\cup I_1\cup I_2$. For simplicity, we denote the extended interval $I\cup I_1\cup I_2$ as $I=(a_1, a_2)$ itself and the extended function $f^i(t)$ as $\lambda^i(t)$.

Thus, if at $t=t_0$, $|\lambda_{i-1}(t_0)|>|\lambda_i(t_0)|\geqslant|\lambda_{i+1}(t_0)|>|\lambda_{i+2}(t_0)|$ and $\lambda_i(t_0)\neq\lambda_{i+1}(t_0)$, then we can define $\lambda^i(t)$ on $I=(a_1, a_2)$ as above and at a_1 and a_2 one of conditions (B)–(D) of Theorem 2.2 hold. When (B) holds at the boundary points then we can further extend the function $\lambda^i(t)$ outside I.

Now let (B) hold at a_1. Then using Theorm 2.3 we can find an open interval I_1 containing a_1 and define a function $\mu^i(t)$ on I_1 such that $\operatorname{Lim}_{t\to a_1}\mu^i(t)=\lambda_i(a_1)=\mu^i(a_1)$. Since $|\lambda_{i-1}(t)|>|\lambda_i(t)|\geqslant|\lambda_{i+1}(t)|>|\lambda_{i+2}(t)|$ for all $t\in I$ we have $\mu^i(t)=\lambda^i(t)$ on $I\cap I_1$. Hence the function $f^i(t)$ defined as

$$f^i(t)=\begin{cases}\lambda^i(t) & \text{on } I,\\ \mu^i(t) & \text{on } I_1,\end{cases}$$

is an extension of $\lambda^i(t)$ on $I\cup I_1$, preserving the continuity and differentiability properties of $\lambda^i(t)$. Again for simplicity we denote the extended function by $\lambda^i(t)$ and the extended interval by $I=(a_1, a_2)$. Now at the boundary points a_1 and a_2 (C) and/or (D) hold.

Extension of I when (C) and/or (D) hold at its boundary points: In this case the eigenvalue to which $\lambda^i(t)$ tends at $t\to a_1$, has multiplicity $\geqslant 2$. Let us assume that $\text{Lim}_{t \searrow a_1} \lambda^i(t) = \lambda_k(a_1)$ and $\lambda_k(a_1)$ has multiplicity n. Then there exist n functions $\lambda^{i_1}(t)$, $\lambda^{i_2}(t), \ldots, \lambda^{i_n}(t)$ such that

$$\operatorname*{Lim}_{t \searrow a_1} \lambda^{i_r}(t) = \lambda_k(a_1), \qquad r = 1, 2, \ldots, n$$

($\lambda^i(t)$ is one of these n functions).

In a similar way there exist n functions $\lambda^{j_1}(t), \ldots, \lambda^{j_n}(t)$ such that

$$\operatorname*{Lim}_{t \nearrow a_1} \lambda^{j_r}(t) = \lambda_k(a_1), \qquad r = 1, 2, \ldots, n.$$

We can consider any one of these $\lambda^{j_r}(t)$, $r = 1, 2, \ldots, n$, as an extension of $\lambda^i(t)$. This extension is obviously continuous at a_1. Now we will prove that if $a_1 \notin B$ (B is the closed, countable set defined as in Section 3, for the symmetric operator $T_t^* T_t$ and its eigenvalues $|\lambda_i(t)|^2$), then the extended function is also differentiable at a_1, whenever $|\lambda_k(t)|^2$ is differentiable at a_1. Since B is countable and hence of zero measure, we do not consider the differentiability of the extended function at a_1, if $a_1 \in B$. Now we prove the assertion.

Let $|\lambda_k(a_1)|^2$ have multiplicity $n_1 \geqslant n$ (since $\lambda_k(a_1)$ has multiplicity n). Since $a_1 \notin B$, from Theorem 3.1, there exists an open interval Z containing a_1 such that $|\lambda_k(t)|^2$ is of multiplicity n_1 for all $t \in Z$. In this case the value of $d/dt |\lambda_l(t)|^2$ at $t = a_1$, if it exists, is equal for all $l = k, k+1, \ldots, k + n_1 - 1$. Then considering any one of the $\lambda^{j_r}(t)$, $r = 1, 2, \ldots, n$, as extension of $\lambda^i(t)$ and denoting the extended function also as $\lambda^i(t)$, we can prove that

$$\frac{d}{dt} \lambda^i(t) = \lambda^i(t)\, |\phi^i(t, t)|^2 \qquad \text{at } t = a_1,$$

as in Theorem 2.3. Since $\lambda^i(a_1) = \lambda_k(a_1)$ has multiplicity n, there exist n orthonormal eigenfunctions $\{\phi^{i,j}(x, a_1)\}_{j=1}^n$ satisfying this formula.

Now we sum up the results proved so far. We considered the FSWH normal operator T_t defined by (1.1) for $0 \leqslant t \leqslant M < \infty$. The spectrum $S_t = \{\lambda_n(t)\}_{n=1}^{\infty}$ of T_t is arranged as

$$|\lambda_1(t)| \geqslant |\lambda_2(t)| \geqslant \cdots \geqslant |\lambda_i(t)| \geqslant \cdots.$$

For a fixed i, $\lambda_i(t)$ is not necessarily a continuous function of t, in general. Starting from an arbitrary point t_0, where $\lambda_i(t_0)$ is simple, we define a new function $\lambda^i(t)$ which takes its values from S_t such that $\lambda^i(t)$ is continuous, almost everywhere differentiable and satisfies (2.10) an $(0, M)$. We can also define $\lambda^i(t)$ in the following way

$$\lambda^i(t) = \lambda_{f_i(t)}(t), \tag{4.25}$$

where $f_t(i)$: $N^+ \to N^+$ (N^+ denotes the set of all nonnegative integers) is a bijective map, and it depends on t.

Proof of Theorem 2.6. The proofs of Theorems 2.1–2.5 given above are derived by modifying the results of FSWH on $L_2[0, t]$ that are self-adjoint. Since these results are also true for general finite section operators, as obtained by Athreya and Vittal Rao [9], it now follows that the above modifications can be carried out for such general finite section normal integral operators also. We omit the details as these are now straightforward.

We conclude this section with a remark on the continuity of eigenfunctions of the normal operator T_t.

If for a fixed i, $\lambda_i(t)$ is simple for all $t \in I$–an open interval, then for each $t \in I$ we can choose a normalized eigenfunction $\phi_i(x, t)$ corresponding to $\lambda_i(t)$ in such a way that $\phi_i(x, t)$ is continuous in t, the continuity being uniform with respect to x. A similar result has been proved in [10] for symmetric integral operators. The same proof holds for normal operators also.

5. Conclusion

For a FSWH operator T_t, using the continuity and differentiability properties of the eigenvalues and the formula (2.3), Vittal Rao, [5], obtained a formula for the Fredholm determinant of T_t which is analogous to the Kac–Akhiezer formula [12, 13]. This was later extended by Mullikin and Vittal Rao [8] for a larger class of self-adjoint integral operators with difference kernels. Athreya and Vittal Rao [9], later obtained a similar result for general self-adjoint integral operators. Using the results of this paper, we shall, in a forthcoming paper, establish similar results for normal integral operators.

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References

1. S. Chandrasekhar, "Radiative Transfer," Dover, New York, 1950.
2. R. Davison, "Neutron Transport Theory," Oxford Univ. Press, London/New York, 1957.

3. D. R. WILLS, Comparison of kinetic theory analyses of linearized Couette flow, *Phys. Fluids* **5** (1962), 127.
4. R. VITTAL RAO, On the eigenvalues of integral operators with difference kernels, *J. Math. Anal. Appl.* **53** (1976), 554–566.
5. R. VITTAL RAO, Extended Akhiezer formula for the Fredholm determinant of difference kernels, *J. Math. Anal. Appl.* **54** (1976), 79–88.
6. R. VITTAL RAO, Relation between Chandrasekhars X-function and the Eigenvalues of integral operators with difference kernels, *J. Math. Anal. Appl.* **59** (1977), 60–68.
7. R. VITTAL RAO, An algebraic system of equations related to the eigenvalues of integral operators with difference kernels, *J. Math. Anal. Appl.* **60** (1977), 379–391.
8. T. W. MULLIKIN AND R. VITTAL RAO, Extended Kac–Akhiezer formula for the Fredholm determinant of integral operators, *J. Math. Anal. Appl.* **61** (1977), 409–415.
9. K. B. ATHREYA AND R. VITTAL RAO, Finite section integral operators—spectral theory and Kac–Akhiezer formula, to appear.
10. R. VITTAL RAO AND N. SUKAVANAM, On the eigenvalues and eigenfunctions of some integral operators, *J. Math. Anal. Appl.* **109** (1985), 463–471.
11. J. A. COCHRAN, "The Analysis of Linear Integral Equations," McGraw–Hill Series in Modern Applied Mathematics, McGraw–Hill, New York, 1972.
12. AKHIEZER, N. I., A continual analogue of some theorems on Toeplitz matrices, *Ukrain. Math. Z.* **16** (1964); *Amer. Math. Soc. Transl.* **50** (1966), 295–316.
13. HIRSCHMANN, I. I., On a formula of Kac and Akhiezer, *J. Math. Mech.* **16** (1966), 167–196.